Mock Exam Geometry

To be discussed on January 12, 2015
Solutions will be posted on Nestor afterwards.

Note: Usage of Do Carmo's textbook is allowed. Give a precise reference to the theory you use for solving the problems. You may *not* use the result of any of the exercises.

Problem 1. (7 + 7 + 7 + 9 = 30 pt.)
Let $\alpha : I \to \mathbb{R}^3$ be a smooth (C^∞) unit-speed curve with positive curvature. $I \subset \mathbb{R}$ is an interval. The curvature and torsion at $\alpha(s)$ are denoted by $\kappa(s)$ and $\tau(s)$, respectively. Let $\overline{\alpha} : I \to \mathbb{R}^3$ be the *offset-curve* given by

$$\overline{\alpha}(s) = \alpha(s) + f(s)\, n(s),$$

where $n(s)$ is the normal of α at $\alpha(s)$ and $f : I \to \mathbb{R}$ is a smooth function. Note that, in general, $\overline{\alpha}$ is *not* a unit-speed curve. Furthermore, it is given that the normal $\overline{n}(s)$ of $\overline{\alpha}$ at $\overline{\alpha}(s)$ is equal to $\pm n(s)$.

1. Express the tangent vector $\overline{\alpha}'(s)$ in the Frenet-Serret frame of α at $\alpha(s)$.
2. Show that $f(s)$ is constant.
3. Prove that the angle between the tangent vector of α at $\alpha(s)$ and the tangent vector of $\overline{\alpha}$ at $\overline{\alpha}(s)$ is constant.
4. Prove that there are constants a and b such that, for $s \in I$:

$$a\kappa(s) + b\tau(s) = 1.$$

Problem 2. (8 + 8 + 7 + 7 = 30 pt.)
Let S be a regular surface in $\mathbb{R}^3$.

1. Prove: If S contains a line, then this line is an asymptotic curve of S.

In the remainder of this assignment S is a one-sheeted hyperboloid of revolution given by $x^2 + y^2 - z^2 = 1$. Furthermore, p is the point $(1, 0, 0)$ of S.

2. Show that the principal curvatures of S at p are equal to 1 and -1, and determine the curvature lines of S through p.
3. Determine both asymptotic directions of S at p, and the corresponding asymptotic curves of S through p.
4. Determine the asymptotic curves through an arbitrary point of S.

Problem 3. $(9 + 9 + 12 = 30$ pt.)
Let C be a regular curve (without self-intersections) in the half-plane $\{(x, 0, z) \mid x > 0\}$. Let S be the surface of revolution in $\mathbb{R}^3$ obtained by rotating C about the z-axis.

1. Which meridians of S are geodesics? Give a proof of your statement(s).

2. Which parallel circles of S are geodesics? Give a proof of your statement(s).

The *angular momentum* of a regular curve $\alpha : \mathbb{R} \to S$ at $\alpha(t)$ is equal to $\alpha(t) \wedge \alpha'(t)$.

3. Prove that the z-component of the angular momentum of a geodesic on S is constant (i.e., independent of t).